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Non-perturbative solution of nonlinear Heisenberg equations

Ladislav Mišta Jr and Radim Filip

Department of Optics, Palacký University, 17. listopadu 50, 772 00 Olomouc, Czech Republic

E-mail: mista@optnw.upol.cz and filip@optnw.upol.cz

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Abstract

A new non-perturbative method of solution of the nonlinear Heisenberg equations in a finite-dimensional subspace is illustrated. The method, being a counterpart of the traditional Schrödinger picture method, is based on a finite operator expansion into the elementary processes. It provides us with insight into the nonlinear quantal interaction from a different point of view. Thus, one can investigate the nonlinear system in both pictures of quantum mechanics.

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1. Introduction

The use of laws of quantum mechanics in the description of nonlinear systems confronts us with qualitatively new difficulties. Namely, to investigate their dynamics in the Heisenberg picture we have to solve the nonlinear operator equations, a task which is highly nontrivial even for the simplest systems. Difficulties are also encountered within the Schrödinger picture once we try to solve the Schrödinger equation explicitly [1]. Both the Schrödinger picture [2,3] and Heisenberg picture [4] methods have been proposed to overcome these difficulties. Since some nonlinear systems solvable analytically in the classical domain become insoluble when they are quantized, one can suppose that they suffer the simultaneous influence of intrinsic stochastic effects, originating from the incompatibility of some observables, and the nonlinearity which makes the behaviour of such systems very complex and thus difficult to describe analytically.

The time evolution of quantum systems can be studied with the help of the widely used Schrödinger picture method based on the integration of a set of linear differential equations for components of a state vector in the Fock basis [5,6]. Unfortunately, the expansion into the Fock-state basis can be infinite for some states, e.g. for a coherent state, yielding an infinite set of these equations. Because it is practically impossible to solve the infinite system of equations, the method provides us with exact solutions only for states from some finite-dimensional subspace of the Hilbert state space. On the other hand, it is sometimes advantageous to calculate the evolution of particular observables in the framework of the Heisenberg picture. The motivation of this paper is to find the operator analogue of the Schrödinger picture method in the Heisenberg picture and to show the equivalence and deep relationship between them.

As an illustrative example we consider here the simple system composed of two harmonic oscillators which oscillate with frequencies ω and 2ω and which are described by the annihilation (creation) operators $\hat{a}_1(\hat{a}_1^\dagger)$ and $\hat{a}_2(\hat{a}_2^\dagger)$ obeying the standard boson-type commutation rules

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \qquad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \qquad i, j = 1, 2. \tag{1}$$

Let the two oscillators interact nonlinearly according to the following interaction Hamiltonian (the second-harmonic generation process [7]):

$$\hat{H} = -\hbar\Gamma\hat{a}_1^{\dagger 2}\hat{a}_2 + \text{H.c.} \tag{2}$$

where Γ denotes the nonlinear coupling constant; the symbol $\hbar$ is the reduced Planck constant and H.c. stands for the Hermitian conjugate. Here and in the following we assume that the free evolution has been eliminated by the appropriate unitary transformation.

Employing the commutation rules (1) one can directly prove the existence of the following integral of motion:

$$\hat{N} = \hat{n}_1 + 2\hat{n}_2 \tag{3}$$

corresponding to the total energy of the system. Here the photon number operator $\hat{n}_j = \hat{a}_j^\dagger\hat{a}_j$ of the jth oscillator, $j = 1, 2$ has been introduced. The eigenvectors of the integral of motion (3) then provide us with the natural orthonormal and complete basis in which the expressions have simple form. They are easy to find and have the form

$$\left\{|N-2l, l\rangle, l = 0, 1, \ldots, \left[\frac{N}{2}\right], N = 0, 1, \ldots\right\} \tag{4}$$

with orthonormality condition

$$\langle N-2l, l|M-2k, k\rangle = \delta_{NM}\delta_{lk} \tag{5}$$

and the resolution of unity operator

$$\sum_{N=0}^{\infty}\sum_{l=0}^{[\frac{N}{2}]}|N-2l, l\rangle\langle N-2l, l| = \hat{1} \tag{6}$$

where $|n_1, n_2\rangle$ is the Fock state having energy $\hbar\omega n_1 + 2\hbar\omega n_2$; N is the eigenvalue of (3), $[N/2]$ represents the greatest integer less or equal to $N/2$ and δ_{lk} is the Kronecker symbol. The Hilbert state space of our system can then be expressed as a direct sum

$$\mathcal{H} = \sum_{N=0}^{\infty}\oplus\mathcal{H}^{(N)} \tag{7}$$

of the invariant $[N/2]$ + 1-dimensional subspaces $\mathcal{H}^{(N)}$ spanned on the basis vectors $|N - 2l, l\rangle, l = 0, 1, \ldots, [N/2]$ corresponding to the fixed eigenvalue N. Using the standard properties of the annihilation and creation operators of the harmonic oscillator

$$\hat{a}_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle \qquad \hat{a}_i^\dagger|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle \qquad i = 1, 2 \tag{8}$$

and employing condition (5) one can show that the Hamiltonian (2) is represented by the following block diagonal matrix:

$$\begin{aligned}\langle N-2l, l|\hat{H}|M-2k, k\rangle = -\hbar[&\Gamma\sqrt{(l+1)(N-2l)(N-2l-1)}\delta_{k,l+1} \\ &+\Gamma^*\sqrt{l(N-2l+2)(N-2l+1)}\delta_{k,l-1}]\delta_{NM}\end{aligned} \tag{9}$$

where the symbol '$*$' represents the complex conjugation.

2. Schrödinger picture

Let us first recall the results obtained with the help of the Schrödinger picture method when applied to our system. As is well known, the time evolution of the state vector is governed by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{10}$$

where Hamiltonian $\hat{H}$ is given in (2). Rewriting (10) into the basis (4) with the help of (6) and (9) we successively arrive at the infinite number of sets of linear differential equations

$$\begin{aligned}\frac{\mathrm{d}C_{N,l}}{\mathrm{d}t} &= \mathrm{i}\Gamma\sqrt{(l+1)(N-2l)(N-2l-1)}C_{N,l+1}\\ &\quad+\mathrm{i}\Gamma^*\sqrt{l(N-2l+2)(N-2l+1)}C_{N,l-1}\end{aligned} \tag{11}$$

for components $C_{N,l} \equiv \langle N-2l, l|\psi(t)\rangle$, where $N = 0, 1, \ldots$ and $l = 0, 1, \ldots, [N/2]$. Assuming, however, the initial state to be from the finite-dimensional subspace

$$\mathcal{H}_K = \sum_{N=0}^{K} \oplus\mathcal{H}^{(N)} \tag{12}$$

it is sufficient to solve only $K + 1$ such sets labelled by eigenvalues $N = 0, 1, \ldots, K$ each of them with $[N/2] + 1$ equations. Particularly, for states belonging to the subspace $\mathcal{H}_2$, the set (11) is of the form

$$\frac{\mathrm{d}C_{0,0}}{\mathrm{d}t} = \frac{\mathrm{d}C_{1,0}}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}C_{2,0}}{\mathrm{d}t} = \mathrm{i}\sqrt{2}\Gamma C_{2,1} \qquad \frac{\mathrm{d}C_{2,1}}{\mathrm{d}t} = \mathrm{i}\sqrt{2}\Gamma^* C_{2,0} \tag{13}$$

and can be solved analytically. An interesting result is obtained assuming the system to be in the state $|0, 1\rangle$ at the beginning of the interaction. The initial conditions for the set (13) are then $C_{0,0}(0) = C_{1,0}(0) = C_{2,0}(0) = 0$, $C_{2,1}(0) = 1$ and the solution of (10) reads as

$$|\psi(t)\rangle = \sum_{N=0}^{2}\sum_{l=0}^{[\frac{N}{2}]} C_{N,l}(t)|N-2l, l\rangle = \mathrm{i}\frac{\Gamma}{|\Gamma|}\sin(\sqrt{2}|\Gamma|t)|2, 0\rangle + \cos(\sqrt{2}|\Gamma|t)|0, 1\rangle. \tag{14}$$

Hence we obtain the following expressions for the mean number of energy quanta in oscillators 1 and 2 in state (14):

$$\langle\hat{n}_1(t)\rangle = 2\sin^2(\sqrt{2}|\Gamma|t) \qquad \langle\hat{n}_2(t)\rangle = \cos^2(\sqrt{2}|\Gamma|t). \tag{15}$$

This nonclassical oscillatory behaviour can be interpreted from the point of view of the Schrödinger picture as being a manifestation of quantum interference effect (14).

3. Heisenberg picture

In this picture the operators $\hat{a}_1$ and $\hat{a}_2$ for the system of interest evolve according to the Heisenberg equations of motion

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{a}_j}{\mathrm{d}t} = [\hat{a}_j, \hat{H}] \qquad j = 1, 2 \tag{16}$$

which after substitution of (2) into (16) and application of (1) read as

$$\frac{\mathrm{d}\hat{a}_1}{\mathrm{d}t} = 2\mathrm{i}\Gamma\hat{a}_1^\dagger\hat{a}_2 \qquad \frac{\mathrm{d}\hat{a}_2}{\mathrm{d}t} = \mathrm{i}\Gamma^*\hat{a}_1^2. \tag{17}$$

It is also well known that the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ can be equivalently expressed as

$$\hat{a}_j(t) = \exp\left(\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\hat{a}_j(0)\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\mathrm{d}^k\hat{a}_j(0)}{\mathrm{d}t^k}t^k \qquad j = 1, 2 \tag{18}$$

where the exponential operators have been expanded and (16) has been used repeatedly.

The power series on the rhs of (18) is a perturbative solution of equations (17) and provides us with two important pieces of information. (It is advantageous to work with the normal ordering of the operators in which all creation operators stand to the left of all annihilation operators.) Firstly, the operator part of solution (18), given by the derivatives of operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ at $t = 0$, cannot contain products of operators other than those leading to the annihilation of one energy quantum from the corresponding oscillator. This can be proved by deriving the Heisenberg equations of motion (17) and using the commutators (1) to obtain the normally ordered expressions. In what follows, any such product of operators at $t = 0$ is called a *process* in the corresponding oscillator and the number of operators in the product is called an *order* of the process. Secondly, calculating the perturbative solution (18) to sufficiently high order and rearranging its terms appropriately, one can see that the solution is of the form of a finite sum of the processes multiplied by various polynomials in t, which can constitute the first few terms of power series of well known functions (this can be verified at least for the first few processes). This different point of view to the standard perturbative solution [8] is the core of our *non-perturbative* method developed in the following text. One can surmise that, going to infinity in the iterative procedure, the solution of the Heisenberg equations of motion (17) is of the form of the *infinite* sum of processes multiplied by some time-dependent functions

$$\hat{a}_1(t) = \hat{a}_1 + f_1(t)\hat{a}_1^\dagger\hat{a}_2 + f_2(t)\hat{a}_1^\dagger\hat{a}_1^2 + f_3(t)\hat{a}_2^\dagger\hat{a}_1\hat{a}_2 + \cdots \tag{19}$$

$$\hat{a}_2(t) = g_1(t)\hat{a}_2 + g_2(t)\hat{a}_1^2 + g_3(t)\hat{a}_1^\dagger\hat{a}_1\hat{a}_2 + g_4(t)\hat{a}_2^\dagger\hat{a}_2^2 + \cdots \tag{20}$$

where $\hat{a}_j \equiv \hat{a}_j(0)$, $j = 1, 2$. The functions f_j and g_j are called *amplitudes* of the corresponding processes in the following text.

Substituting (19) and (20) into (17) and comparing the coefficients related to the same process, the amplitudes f_j and g_j can be determined as solutions of a system of ordinary differential equations. For example, the equations for amplitudes f_1 and g_1 together with the initial conditions read

$$\frac{\mathrm{d}}{\mathrm{d}t}f_1(t) = 2\mathrm{i}\Gamma g_1(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}g_1(t) = \mathrm{i}\Gamma^* f_1(t) \qquad f_1(0) = 0 \quad g_1(0) = 1 \tag{21}$$

and have the following solutions:

$$f_1(t) = \mathrm{i}\frac{\sqrt{2}\Gamma}{|\Gamma|}\sin(\sqrt{2}|\Gamma|t) \qquad g_1(t) = \cos(\sqrt{2}|\Gamma|t). \tag{22}$$

Employing (19) and (20), the operators of the number of the energy quanta in the oscillators 1 and 2 are of the form

$$\hat{n}_1(t) = \hat{a}_1^\dagger\hat{a}_1 + 2\sin^2(\sqrt{2}|\Gamma|t)\hat{a}_2^\dagger\hat{a}_2 + \cdots \tag{23}$$

$$\hat{n}_2(t) = \cos^2(\sqrt{2}|\Gamma|t)\hat{a}_2^\dagger\hat{a}_2 + \cdots \tag{24}$$

where relations (1) and (22) have been used. It is worth noting that, contrary to the rhs of (24), the second term in (23) is the quantum contribution originating from the commutator $[\hat{a}_1, \hat{a}_1^\dagger] = \hat{1}$. Considering as in the previous section the input state to be $|0, 1\rangle$ state and assuming that the terms represented by dots in (23) and (24) do not contribute, one obtains for the mean number of the energy quanta in the oscillators the expressions

$$\langle\hat{n}_1(t)\rangle = |f_1(t)|^2 = 2\sin^2(\sqrt{2}|\Gamma|t) \qquad \langle\hat{n}_2(t)\rangle = |g_1(t)|^2 = \cos^2(\sqrt{2}|\Gamma|t) \tag{25}$$

which are identical with the results (15) obtained by means of the Schrödinger picture method. Note that this derivation illustrates not only the mathematical equivalence of both methods but also their difference when one tries to distinguish between the classical and quantum contributions.

Although one might view the method just described as being a satisfactory method, its conclusion (25) rests on two crucial assumptions which are not justified at all. Firstly, we have assumed implicitly, when deriving (21), that the higher-order processes do not affect the first-order ones (consequently we have obtained the finite set of differential equations for amplitudes f_1 and g_1). Secondly, the mean numbers of energy quanta in state $|0, 1\rangle$ given by (25) have been derived under the assumption that only explicitly given terms in (23) and (24) contribute. To show that this is really the case, we have to formalize and detail the Heisenberg picture method. This is done in the following section.

4. General method

The previous section gave an illustrative example of how one can treat the system (2) within the framework of Heisenberg picture on the intuitive basis. In this section we try to justify the intuitive assumptions discussed above and to generalize this treatment to the arbitrary finite-dimensional subspace. This can be achieved by suitable parametrization of the problem under discussion. Let us rewrite the expansions (19) and (20) in compact form:

$$\hat{a}_1(t) = \sum_{i,j,k,l=0}^{\infty} f_{ijkl}(t)(\hat{a}_1^\dagger)^i(\hat{a}_2^\dagger)^j \hat{a}_1^k \hat{a}_2^l \tag{26}$$

$$\hat{a}_2(t) = \sum_{o,p,r,s=0}^{\infty} g_{oprs}(t)(\hat{a}_1^\dagger)^o(\hat{a}_2^\dagger)^p \hat{a}_1^r \hat{a}_2^s \tag{27}$$

where

$$2l + k - 2j - i = 1 \tag{28}$$

and

$$2s + r - 2p - o = 2 \tag{29}$$

holds. There are two facts which can make the convenient parametrization easier to find. Firstly, as in the Schrödinger picture we can employ the existence of the integral of motion (3). Secondly, the discussion in the previous section indicates that the order of the process is of importance. Therefore, we put

$$N = k + 2l \qquad M = i + 2j \qquad m = k + l \qquad R = i + j + k + l \tag{30}$$

for oscillator 1 and, similarly,

$$N = r + 2s \qquad M = o + 2p \qquad m = r + s \qquad R = o + p + r + s \tag{31}$$

for oscillator 2. Thus, each process is parametrized by the parameters $N(M)$ (representing the amount of the annihilated (created) energy in the process), m (the total number of the annihilated energy quanta) and R (the order of the process). Substituting (30) into (26) and eliminating N by means of (28) we arrive at the following expansion:

$$\hat{a}_1(t) = \sum_{M=0}^{\infty} \sum_{m=[\frac{M}{2}]+1}^{M+1} \sum_{R=[\frac{M+1}{2}]+m}^{M+m} f_{MmR}(t)(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}. \tag{32}$$

In the same way we obtain

$$\hat{a}_2(t) = \sum_{M=0}^{\infty} \sum_{m=[\frac{M+1}{2}]+1}^{M+2} \sum_{R=[\frac{M+1}{2}]+m}^{M+m} g_{MmR}(t)(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-2}\hat{a}_2^{M-m+2}. \tag{33}$$

Substituting (32) and (33) into the Heisenberg equations of motion (17), putting all occurring operators in normal order and comparing the terms corresponding to the same processes, one obtains the set of differential equations for amplitudes f_{MmR} and g_{MmR}. Although the set of equations seems to be infinite at first sight, in fact the opposite is true. A more detailed analysis reveals that for each pair (M, m) there is a set of $M+1$ linear differential equations for amplitudes $f_{MmR'}$, $R' = [\frac{M+1}{2}]+m, \ldots, M+m$ and $g_{M-1mR''}$, $R'' = [\frac{M}{2}]+m, \ldots, M+m-1$. Here and in the following the single-primed and double-primed letters are used in order to distinguish between the parameters related to oscillator 1 and 2. The other terms in the rhs of the set of equations can be found by solving the independent set of equations corresponding to $M-1$ and play the role of known variable coefficients and source terms. Hence it is evident, among other things, that for fixed M the processes in oscillator 1 for which $M' > M$ and the processes in oscillator 2 for which $M'' > M-1$ do not affect the processes corresponding to $M' \leqslant M$ and $M'' \leqslant M-1$, as we wanted to prove. Details of these calculations and the explicit form of the set of equations are given in appendix A.

As an illustrative example we give explicitly the sets of equations corresponding to pairs $(0, 1)$, $(1, 1)$ and $(1, 2)$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{011}(t) = 0 \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{112}(t) = 2\mathrm{i}\Gamma f_{011}^*(t) g_{011}(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t} g_{011}(t) = \mathrm{i}\Gamma^* f_{011}(t) f_{112}(t) \tag{35}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{123}(t) = 2\mathrm{i}\Gamma f_{011}^*(t) g_{022}(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t} g_{022}(t) = \mathrm{i}\Gamma^* f_{011}(t)[f_{011}(t) + f_{123}(t)] \tag{36}$$

where the initial conditions read $f_{011}(0) = g_{011}(0) = 1$, $f_{112}(0) = f_{123}(0) = g_{022}(0) = 0$. Their solutions are easy to find using the standard methods and are of the form

$$\begin{aligned} &f_{011}(t) = 1 \\ &f_{112}(t) = \mathrm{i}\frac{\sqrt{2}\Gamma}{|\Gamma|}\sin(\sqrt{2}|\Gamma|t) \qquad g_{011}(t) = \cos(\sqrt{2}|\Gamma|t) \\ &f_{123}(t) = \cos(\sqrt{2}|\Gamma|t) - 1 \qquad g_{022}(t) = \mathrm{i}\frac{\Gamma^*}{\sqrt{2}|\Gamma|}\sin(\sqrt{2}|\Gamma|t). \end{aligned} \tag{37}$$

Before going further let us note that, since the infinite series (32) and (33) with amplitudes being the solutions of the corresponding sets of equations satisfy the Heisenberg equations of motion (17) identically, the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ preserve the commutation rules (1).

One might still object that our method cannot be used in practice since we are not able to calculate all the amplitudes in infinite series (32) and (33). This difficulty is, however, overcome if one considers the following fact. Calculating the matrix element of the process corresponding to parameters M, m and R for oscillator 1,

$$\langle N_1 - 2l_1, l_1|(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}|N_2 - 2l_2, l_2\rangle \tag{38}$$

it can be shown (see appendix B) that it does not vanish only if the following inequalities are satisfied:

$$N_1 \geqslant M \qquad N_2 \geqslant M+1. \tag{39}$$

Repeating the same discussion for the same matrix element of the process in oscillator 2 characterized by the parameters M, m and R one arrives at

$$N_1 \geqslant M \qquad N_2 \geqslant M + 2. \tag{40}$$

The inequalities (39) and (40) can be interpreted as follows. Restricting ourselves to the finite-dimensional subspace $\mathcal{H}_K$ of the whole Hilbert space (7) only processes in oscillator 1 (2) for which $M' = 0, 1, \ldots, K-1$ ($M'' = 0, 1, \ldots, K-2$) are represented by nonzero matrices. In other words, the time evolution of the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ on the subspace $\mathcal{H}_K$ is known once the amplitudes $f_{M'm'R'}$, $M' = 0, 1, \ldots, K-1$; $m' = [\frac{M'}{2}] + 1, \ldots, M'+1$; $R' = [\frac{M'+1}{2}] + m', \ldots, M' + m'$ and $g_{M''m''R''}$, $M'' = 0, 1, \ldots, K-2$; $m'' = [\frac{M''+1}{2}] + 1, \ldots, M''+2$; $R'' = [\frac{M''+1}{2}] + m'', \ldots, M'' + m''$ are determined. This requires sequential solution of $[\frac{M}{2}] + 1$ sets of $M = 0, 1, \ldots, K$ differential equations for these amplitudes.

Since the series (32) and (33) are terminated naturally when considering only finite-dimensional subspace $\mathcal{H}_K$, a natural question arises: whether the last-named amplitudes determine not only the evolution of operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ but also the evolution of the mean value of any operator in the subspace $\mathcal{H}_K$. Now we will prove that this is really the case.

It is well known that any operator at time t in the space $\mathcal{H}$ can be expressed as a sum of the normally ordered products $(\hat{a}_1^\dagger)^i(t)(\hat{a}_2^\dagger)^j(t)\hat{a}_1^k(t)\hat{a}_2^l(t)$. Hence, it is sufficient to prove the statement for these products only. The commutation rules (1) enable us to show that

$$\begin{aligned}&\hat{N}(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}|N-2l, l\rangle \\ &\qquad = (N-1)(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}|N-2l, l\rangle\end{aligned} \tag{41}$$

for oscillator 1 and, similarly,

$$\begin{aligned}&\hat{N}(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-2}\hat{a}_2^{M-m+2}|N-2l, l\rangle \\ &\qquad = (N-2)(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-2}\hat{a}_2^{M-m+2}|N-2l, l\rangle\end{aligned} \tag{42}$$

for oscillator 2. Consequently,

$$\hat{N}\hat{a}_j(t)|N-2l, l\rangle = (N-j)\hat{a}_j(t)|N-2l, l\rangle \qquad j = 1, 2 \tag{43}$$

as one can verify, using (32) and (33). From that it follows that $\hat{a}_j(t)\mathcal{H}_K \subset \mathcal{H}_{K-j}$, $j = 1, 2$. Since the first annihilation (creation) operator to the right (left) in the matrix elements

$$\langle N_1 - 2l_1, l_1|(\hat{a}_1^\dagger)^i(t)(\hat{a}_2^\dagger)^j(t)\hat{a}_1^k(t)\hat{a}_2^l(t)|N_2 - 2l_2, l_2\rangle \qquad N_1, N_2 = 0, 1, \ldots, K \tag{44}$$

transforms the basis vector to the right (left) into the subspace embedded into $\mathcal{H}_K$, the series (32) and (33) for the following annihilation (creation) operators must terminate even further than those for the first annihilation (creation) operator. Therefore, no other amplitudes except for those which determine the time evolution of operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ on the subspace $\mathcal{H}_K$ can appear in the expression (44). This is what we wanted to prove.

Summarizing the previous results one can see that, in order to determine the time evolution of the mean value of any operator in a finite-dimensional subspace of space (7), one has to solve only a *finite* number of linear differential equations for amplitudes.

To illustrate this, let us restrict our attention to subspace $\mathcal{H}_2$. The mean value of any operator in the subspace then can be calculated by means of the following truncated series:

$$\hat{a}_1(t) = f_{011}(t)\hat{a}_1 + f_{112}(t)\hat{a}_1^\dagger\hat{a}_2 + f_{123}(t)\hat{a}_1^\dagger\hat{a}_1^2 \tag{45}$$

$$\hat{a}_2(t) = g_{011}(t)\hat{a}_2 + g_{022}(t)\hat{a}_1^2 \tag{46}$$

where the amplitudes in the rhs are given in (37). The operators of the number of energy quanta in oscillators 1 and 2 at time t in the subspace are then of the form

$$\begin{aligned}\hat{n}_1(t) &= [f_{112}(f_{011}^* + f_{123}^*)](\hat{a}_1^\dagger)^2\hat{a}_2 + f_{011}f_{123}^*(\hat{a}_1^\dagger)^2\hat{a}_1^2 + f_{112}f_{123}^*(\hat{a}_1^\dagger)^3\hat{a}_1\hat{a}_2 + \text{H.c.}] \\ &\quad + |f_{011}|^2\hat{n}_1 + |f_{112}|^2(\hat{n}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2) + |f_{123}|^2[(\hat{a}_1^\dagger)^2\hat{a}_1^2 + (\hat{a}_1^\dagger)^3\hat{a}_1^3] \\ \hat{n}_2(t) &= |g_{011}|^2\hat{n}_2 + |g_{022}|^2(\hat{a}_1^\dagger)^2\hat{a}_1^2 + [g_{011}g_{022}^*(\hat{a}_1^\dagger)^2\hat{a}_2 + \text{H.c.}]\end{aligned} \tag{47}$$

where the time arguments in the rhs were omitted for simplicity. Calculating now the mean numbers of the energy quanta in the oscillators in state $|0, 1\rangle \in \mathcal{H}_2$ using (37) and (47) one obtains exactly the same results as those of given by equations (15). Now we are sure, however, that no contributions were omitted in the course of calculation of the mean values.

Although the truncated series for operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ in a finite-dimensional subspace cover all operations within the subspace, one has to handle them carefully when calculating the mean values of other than normally ordered operators. The following simple example illustrates this. Substituting (37) into (46) one arrives at

$$[\hat{a}_2(t), \hat{a}_2^\dagger(t)] = \hat{1} + 2\sin^2(\sqrt{2}|\Gamma|t)\hat{a}_1^\dagger\hat{a}_1 \neq \hat{1}. \tag{48}$$

In the same way one can prove that $[\hat{a}_1(t), \hat{a}_1^\dagger(t)] \neq \hat{1}$. The inconsistency of the commutators is a consequence of the fact that the antinormally ordered products $\hat{a}_j(t)\hat{a}_j^\dagger(t)$, $j = 1, 2$ generate states outside the subspace. Therefore, in order to obtain the correct mean value for an operator which is not in normal order, one has to put it in normal order first using the correct commutation rules $[\hat{a}_j(t), \hat{a}_j^\dagger(t)] = \hat{1}$, $j = 1, 2$ valid for infinite series (32) and (33). Having the operator expressed by means of normally ordered moments one can then substitute the truncated series instead of the infinite series and calculate the sought mean value.

5. Conclusion

On a simple example we illustrate how to solve the nonlinear Heisenberg equations of motion in a finite-dimensional subspace using the finite expansion of annihilation operators into the sum of the elementary processes. The idea of the method is not restricted to this example and provides us with a recipe to treat other nonlinear interactions. The time evolution of any operator in the subspace is then governed by a finite set of the linear c-number differential equations for amplitudes. Due to the hierarchy of the processes, the set of equations splits into several subsets which can be solved step by step. Thus, the problem of solution of the q-number Heisenberg equations is transformed into finding the solution of the finite sets of linear c-number differential equations, which can be handled numerically. This gives a nice interpretation and deeper insight into what happens in the course of the nonlinear quantal interaction in the language of elementary processes. It also enables us to identify the nonclassical contributions. This instructive interpretation cannot be obtained within the framework of the Schrödinger picture.

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Appendix A

In order to find the differential equations for amplitudes f_{MmR}, $M = 0, 1, \ldots$; $m = [\frac{M}{2}] + 1, \ldots, M+1$; $R = [\frac{M+1}{2}]+m, \ldots, M+m$ and g_{MmR}, $M = 0, 1, \ldots$; $m = [\frac{M+1}{2}]+1, \ldots, M+2$; $R = [\frac{M+1}{2}]+m, \ldots, M+m$ we must first substitute expressions (32) and (33) into both sides of Heisenberg equations of motion (17). Putting the operator products in the rhs of the equations in normal order with the help of formula [8]

$$\hat{a}_i^m(\hat{a}_i^\dagger)^n = \sum_{j=0}^{\min(m,n)} j!\binom{m}{j}\binom{n}{j}(\hat{a}_i^\dagger)^{n-j}\hat{a}_i^{m-j} \qquad i = 1, 2 \tag{49}$$

and comparing the expressions corresponding to the same processes, we obtain the sought set of differential equations for amplitudes:

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{MmR}(t) = 2\mathrm{i}\Gamma \sum_{M_1,M_2=0}^{\infty} \sum_{m_1=[\frac{M_1}{2}]+1}^{M_1+1} \sum_{m_2=[\frac{M_2+1}{2}]+1}^{M_2+2} \sum_{R_1=[\frac{M_1+1}{2}]+m_1}^{M_1+m_1} \sum_{R_2=[\frac{M_2+1}{2}]+m_2}^{M_2+m_2} \times \binom{2R_1 - M_1 - 2m_1}{s_1}\binom{2R_2 - M_2 - 2m_2}{s_1}\binom{M_1 + m_1 - R_1}{s_2} \times \binom{M_2 + m_2 - R_2}{s_2} s_1! s_2! f^*_{M_1 m_1 R_1}(t) g_{M_2 m_2 R_2}(t) \tag{50}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{MmR}(t) = \mathrm{i}\Gamma^* \sum_{M_1,M_2=0}^{\infty} \sum_{m_1=[\frac{M_1}{2}]+1}^{M_1+1} \sum_{m_2=[\frac{M_2}{2}]+1}^{M_2+1} \sum_{R_1=[\frac{M_1+1}{2}]+m_1}^{M_1+m_1} \sum_{R_2=[\frac{M_2+1}{2}]+m_2}^{M_2+m_2} \times \binom{2m_1 - M_1 - 1}{s_1}\binom{2R_2 - M_2 - 2m_2}{s_1}\binom{M_1 - m_1 + 1}{s_2} \times \binom{M_2 + m_2 - R_2}{s_2} s_1! s_2! f_{M_1 m_1 R_1}(t) f_{M_2 m_2 R_2}(t) \tag{51}$$

where

$$\begin{aligned} R_2 &= R + R_1 - 2m - 2m_1 + 2m_2 \\ s_1 &= M - M_1 - M_2 - 2m - 2m_1 + 2m_2 + 2R_1 - 1 \\ s_2 &= M_1 + M_2 - M - m - m_1 + m_2 + R - R_2 + 1 \end{aligned} \tag{52}$$

for equation (50) and

$$\begin{aligned} R_2 &= R - R_1 - 2m + 2m_1 + 2m_2 \\ s_1 &= M - M_1 - M_2 + 2m - 2m_1 - 2m_2 - 2R + 2R_1 + 2R_2 \\ s_2 &= M_1 + M_2 - M - m + m_1 + m_2 + R - R_1 - R_2 \end{aligned} \tag{53}$$

for equation (51). The initial conditions for equations (50) and (51) are $f_{011}(0) = g_{011}(0) = 1$, $f_{MmR}(0) = g_{MmR}(0) = 0$ in all other cases.

The structure of the rhs of (50) reveals that there is a nonzero contribution to the rhs only if the following inequalities hold simultaneously:

$$\begin{aligned} &2R_1 - M_1 - 2m_1 \geqslant s_1 \qquad 2R_2 - M_2 - 2m_2 \geqslant s_1 \\ &M_1 + m_1 - R_1 \geqslant s_2 \qquad M_2 + m_2 - R_2 \geqslant s_2. \end{aligned} \tag{54}$$

Combining this with (52) one finally obtains

$$M_1 \leqslant M - 1 \qquad M_2 \leqslant M - 1. \tag{55}$$

Analogously, rhs of (51) contains nonzero contribution only if

$$\begin{aligned} &2m_1 - M_1 - 1 \geqslant s_1 \qquad 2R_2 - M_2 - 2m_2 \geqslant s_1 \\ &M_1 - m_1 + 1 \geqslant s_2 \qquad M_2 + m_2 - R_2 \geqslant s_2 \end{aligned} \tag{56}$$

hold simultaneously. Consequently, the rhs of (51) cannot contain amplitudes other than those for which

$$M_1 \leqslant M \qquad M_2 \leqslant M + 1. \tag{57}$$

From the inequalities (55) and (57), it follows that for fixed M we have only the finite set of equations (50), (51) for amplitudes $f_{M'm'R'}$, $M' = 0, 1, \ldots, M$; $m' = [\frac{M'}{2}]+1, \ldots, M'+1$; $R' = [\frac{M'+1}{2}] + m', \ldots, M' + m'$ and $g_{M''m''R''}$, $M'' = 0, 1, \ldots, M - 1$; $m'' = [\frac{M''+1}{2}] + 1, \ldots, M'' + 2$;

$R'' = [\frac{M''+1}{2}] + m'', \ldots, M'' + m''$. In other words, the processes in oscillators 1 and 2 parametrized by $M' > M$ and $M'' > M - 1$ do not affect the processes in oscillators 1 and 2 for which $M' \leqslant M$ and $M'' \leqslant M - 1$.

Discussion of the structure of the set of equations (50), (51) can be taken even further. Since the amplitudes $f_{M'm'R'}$, $M' < M$ and $g_{M''m''R''}$, $M'' < M - 1$ can be calculated solving the independent set of equations (50), (51) corresponding to $M - 1$, in fact only the amplitudes $f_{Mm'R'}$, $m' = [\frac{M}{2}] + 1, \ldots, M + 1$; $R' = [\frac{M+1}{2}] + m', \ldots, M + m'$ and $g_{M-1m''R''}$, $m'' = [\frac{M}{2}] + 1, \ldots, M + 1$; $R'' = [\frac{M}{2}] + m'', \ldots, M + m'' - 1$ are mutually coupled. The amplitudes $f_{M'm'R'}$, $M' < M$ and $g_{M''m''R''}$, $M'' < M - 1$ then play the role of known coefficients and source terms and the set of differential equations corresponding to M is linear. Moreover, taking M and m fixed, substituting (52) into (54) and putting $M_1 = M_2 = M - 1$ one obtains the following equality:

$$m_2 = m. \tag{58}$$

Since the same equality can be proved by substituting (53) into (56) and putting $M_1 = M$, $M_2 = M + 1$, one can conclude that only amplitudes $f_{MmR'}$, $R' = [\frac{M+1}{2}] + m, \ldots, M + m$ and $g_{M-1mR''}$, $R'' = [\frac{M}{2}] + m, \ldots, M + m - 1$ are coupled. Hence, for given M and m one has to solve the set of $M + 1$ differential equations (50) and (51).

Appendix B

In this appendix we want to show that the matrix element (38) does not vanish only if the inequalities (39) are satisfied. If we calculate the matrix element (38) with the help of (5) and (8)

$$\begin{aligned} &\langle N_1 - 2l_1, l_1 | (\hat{a}_1^\dagger)^{2R-M-2m} (\hat{a}_2^\dagger)^{M+m-R} \hat{a}_1^{2m-M-1} \hat{a}_2^{M-m+1} | N_2 - 2l_2, l_2 \rangle \\ &\quad = \sqrt{\frac{l_1! l_2! (N_1 - 2l_1)! (N_2 - 2l_2)!}{(N_1 - 2l_1 + M + 2m - 2R)! (N_2 - 2l_2 + M - 2m + 1)!}} \\ &\quad \times \frac{\delta_{N_1-2l_1+4m, N_2-2l_2+2R+1} \delta_{l_1+R+1, l_2+2m}}{\sqrt{(l_1 - M - m + R)! (l_2 - M + m - 1)!}} \end{aligned} \tag{59}$$

it is evident that it does not vanish only if the following inequalities are satisfied simultaneously:

$$\begin{aligned} &N_1 - 2l_1 \geqslant 2R - M - 2m \qquad 2l_1 \geqslant 2M + 2m - 2R \\ &N_2 - 2l_2 \geqslant 2m - M - 1 \qquad 2l_2 \geqslant 2M - 2m + 2. \end{aligned} \tag{60}$$

Hence

$$N_1 \geqslant M \qquad N_2 \geqslant M + 1. \tag{61}$$

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